

NATURAL CONNECTIONS WITH TOTALLY SKEW-SYMMETRIC TORSION ON MANIFOLDS WITH ALMOST CONTACT 3-STRUCTURE AND METRICS OF HERMITIAN-NORDEN TYPE

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ABSTRACT. It is considered a differentiable manifold equipped with a pseudo-Riemannian metric and an almost contact 3-structure so that an almost contact metric structure and two almost contact B-metric structures are generated. There are introduced the so-called associated Nijenhuis tensors for the studied structures. It is given a geometric interpretation of the vanishing of these tensors as a necessary and sufficient condition for the existence of linear connections with totally skew-symmetric torsions preserving the structure. An example of a 7-dimensional manifold with connections of the considered type is given.

1. INTRODUCTION

It is known the notion of an *almost contact 3-structure* on a differentiable manifold of dimension $4n + 3$ ([10, 21]). The product of a manifold with almost contact 3-structure and a real line admits an *almost hypercomplex structure* (cf. [10, 1]).

It is only considered the case of equipping of such a manifold with a Riemannian metric compatible with each of the three structures in the given almost contact 3-structure. This is the so-called *almost contact metric 3-structure*.

In [17], we have introduced a pseudo-Riemannian metric which has another kind of compatibility with the triad of almost contact structures on a manifold with almost contact 3-structure. The product of this manifold of new type and a real line is a $(4n + 4)$ -dimensional manifold which admits an almost hypercomplex structure (J_1, J_2, J_3) and a Hermitian-Norden metric (briefly, an HN-metric), i.e. J_1 (resp., J_2 and J_3) acts as an isometry (resp., act as anti-isometries) with respect to the pseudo-Riemannian metric of neutral signature in each tangent fibre. This structure is called an *almost hypercomplex HN-metric structure* and it is studied in [8, 11, 13, 18], etc. The constructed structure on $(4n + 3)$ -dimensional manifolds we call an *almost contact 3-structure with metrics of Hermitian-Norden type* (briefly, an HN-type).

The goal of the present paper is to introduce an appropriate tensor on a manifold with almost contact 3-structure and metrics of HN-type such that the vanishing of this tensor is a necessary and sufficient condition for existence of linear connections with totally skew-symmetric torsion preserving the almost contact 3-structure and the metric of HN-type.

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Convention 1. Let \mathcal{M} be an almost contact manifold and $\mathcal{M} \times \mathbb{R}$ be the corresponding almost complex manifold.

- (i) We shall use x, y, z, \dots to denote smooth vector fields on \mathcal{M} , i.e. $x, y, z \in \mathfrak{X}(\mathcal{M})$, or vectors in the tangent space $T_p\mathcal{M}$ at $p \in \mathcal{M}$;
- (ii) We shall use X, Y, Z, \dots to denote smooth vector fields on $\mathcal{M} \times \mathbb{R}$ or tangent vectors in $T_{\bar{p}}(\mathcal{M} \times \mathbb{R})$ at $\bar{p} \in \mathcal{M} \times \mathbb{R}$.

2. MANIFOLDS WITH ALMOST CONTACT HN-METRIC 3-STRUCTURE

Let $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $(\alpha = 1, 2, 3)$ be a manifold with an almost contact 3-structure, i.e. \mathcal{M} is a $(4n+3)$ -dimensional differentiable manifold with three almost contact structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $(\alpha = 1, 2, 3)$ consisting of endomorphisms φ_α of the tangent bundle, Reeb vector fields ξ_α and their dual contact 1-forms η_α satisfying the following identities:

$$(1) \quad \begin{aligned} \varphi_\alpha \circ \varphi_\beta &= -\delta_{\alpha\beta} I + \xi_\alpha \otimes \eta_\beta + \epsilon_{\alpha\beta\gamma} \varphi_\gamma, \\ \varphi_\alpha \xi_\beta &= \epsilon_{\alpha\beta\gamma} \xi_\gamma, \quad \eta_\alpha \circ \varphi_\beta = \epsilon_{\alpha\beta\gamma} \eta_\gamma, \quad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \end{aligned}$$

where $\alpha, \beta, \gamma \in \{1, 2, 3\}$, I is the identity on the algebra $\mathfrak{X}(\mathcal{M})$, $\delta_{\alpha\beta}$ is the Kronecker delta, $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol, i.e. either the sign of the permutation (α, β, γ) of $(1, 2, 3)$ or 0 if any index is repeated.

In [17], we introduce the following notion. A pseudo-Riemannian metric g is called a *metric of Hermitian-Norden type* (in short an *HN-metric*) on a manifold with almost contact 3-structure $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha)$, if it satisfies the identities

$$(2) \quad g(\varphi_\alpha x, \varphi_\alpha y) = \varepsilon_\alpha g(x, y) + \eta_\alpha(x) \eta_\alpha(y), \quad \alpha = 1, 2, 3$$

for some cyclic permutation $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of $(1, -1, -1)$. We suppose for the sake of definiteness that

$$\varepsilon_\alpha = \begin{cases} 1, & \alpha = 1; \\ -1, & \alpha = 2, 3. \end{cases}$$

Then, $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ we call an *almost contact HN-metric 3-structure*.

Actually, this 3-structure consists of an almost contact metric structure for $\alpha = 1$ and two almost contact B-metric structures for $\alpha = 2$ and $\alpha = 3$. Then g is a compatible metric with respect to $(\varphi_1, \xi_1, \eta_1, g)$ and g is a B-metric with respect to $(\varphi_2, \xi_2, \eta_2, g)$ and $(\varphi_3, \xi_3, \eta_3, g)$ on \mathcal{M} .

The fundamental tensors of a manifold with almost contact HN-metric 3-structure are the three $(0, 3)$ -tensors determined by

$$(3) \quad F_\alpha(x, y, z) = g((\nabla_x \varphi_\alpha) y, z), \quad \alpha = 1, 2, 3,$$

where ∇ is the Levi-Civita connection generated by g . They have the following basic properties caused by the structures

$$(4) \quad \begin{aligned} F_\alpha(x, y, z) &= -\varepsilon_\alpha F_\alpha(x, z, y) \\ &= -\varepsilon_\alpha F_\alpha(x, \varphi_\alpha y, \varphi_\alpha z) + F_\alpha(x, \xi_\alpha, z) \eta_\alpha(y) \\ &\quad + F_\alpha(x, y, \xi_\alpha) \eta_\alpha(z). \end{aligned}$$

Bearing in mind the following consequence of (2)

$$(5) \quad \eta_\alpha = -\varepsilon_\alpha \xi_\alpha \lrcorner g,$$

as well as (3), we have the following relations

$$(6) \quad F_\alpha(x, \varphi_\alpha y, \xi_\alpha) = -\varepsilon_\alpha (\nabla_x \eta_\alpha)(y) = g(\nabla_x \xi_\alpha, y).$$

Let $\mathfrak{L}_{\xi_\alpha} g$ denote the Lie derivative of g along ξ_α . We have the following relations using (2)

$$(7) \quad \begin{aligned} (\mathfrak{L}_{\xi_\alpha} g)(x, y) &= g(\nabla_x \xi_\alpha, y) + g(x, \nabla_y \xi_\alpha) \\ &= -\varepsilon_\alpha((\nabla_x \eta_\alpha)(y) + (\nabla_y \eta_\alpha)(x)). \end{aligned}$$

We use the known classifications of the almost contact metric manifolds and the almost contact B-metric manifolds in terms of F_α given in [2] and [6], respectively. The former classification is relevant for $\alpha = 1$ and contains 12 basic classes \mathcal{W}_i ($i = 1, 2, \dots, 12$), whereas the latter one consists of 11 basic classes \mathcal{F}_i ($i = 1, 2, \dots, 11$) and it applies for $\alpha = 2$ or $\alpha = 3$.

3. ASSOCIATED NIJENHUIS TENSORS ON MANIFOLDS WITH ALMOST CONTACT HN-METRIC 3-STRUCTURE

As it is known, for each $\alpha \in \{1, 2, 3\}$ the Nijenhuis tensor N_α of an almost contact manifold $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha)$ is defined by:

$$N_\alpha = [\varphi_\alpha, \varphi_\alpha] + \xi_\alpha \otimes d\eta_\alpha,$$

where we have $[\varphi_\alpha, \varphi_\alpha](x, y) = \varphi_\alpha^2[x, y] + [\varphi_\alpha x, \varphi_\alpha y] - \varphi_\alpha[\varphi_\alpha x, y] - \varphi_\alpha[x, \varphi_\alpha y]$ and $d\eta_\alpha(x, y) = (\nabla_x \eta_\alpha)y - (\nabla_y \eta_\alpha)x$. Moreover, let us recall, if two almost contact structures in an almost contact 3-structure are normal, then the third one is also normal [10, 22].

Let us consider the symmetric braces $\{x, y\}$ introduced by the following equalities for a pseudo-Riemannian metric g

$$(8) \quad \begin{aligned} g(\{x, y\}, z) &= g(\nabla_x y + \nabla_y x, z) \\ &= xg(y, z) + yg(x, z) - zg(x, y) - g([y, z], x) + g([z, x], y). \end{aligned}$$

For the almost contact structure $(\varphi_1, \xi_1, \eta_1)$ and the metric g , we define a symmetric tensor \hat{N}_1 by

$$(9) \quad \hat{N}_1 = \{\varphi_1, \varphi_1\} - \xi_1 \otimes \mathfrak{L}_{\xi_1} g,$$

where $\{\varphi_1, \varphi_1\}$ is the symmetric tensor field of type (1, 2) given by

$$(10) \quad \{\varphi_1, \varphi_1\}(x, y) = \{\varphi_1 x, \varphi_1 y\} + (\varphi_1)^2\{x, y\} - \varphi_1\{\varphi_1 x, y\} - \varphi_1\{x, \varphi_1 y\}.$$

We call \hat{N}_1 an *associated Nijenhuis tensor* on $(\mathcal{M}, \varphi_1, \xi_1, \eta_1, g)$.

The corresponding tensors of type (0, 3) for N_1 and \hat{N}_1 are given by $N_1(x, y, z) = g(N_1(x, y), z)$ and $\hat{N}_1(x, y, z) = g(\hat{N}_1(x, y), z)$, respectively.

By direct consequences of the definitions, we get that N_1 , \hat{N}_1 and $\mathfrak{L}_{\xi_1} g$ are expressed in terms of F_1 as follows:

$$(11) \quad \begin{aligned} N_1(x, y, z) &= F_1(\varphi_1 x, y, z) + F_1(x, y, \varphi_1 z) + F_1(x, \varphi_1 y, \xi_1) \eta_1(z) \\ &\quad - F_1(\varphi_1 y, x, z) - F_1(y, x, \varphi_1 z) - F_1(y, \varphi_1 x, \xi_1) \eta_1(z), \end{aligned}$$

$$(12) \quad \begin{aligned} \hat{N}_1(x, y, z) &= F_1(\varphi_1 x, y, z) + F_1(x, y, \varphi_1 z) + F_1(x, \varphi_1 y, \xi_1) \eta_1(z) \\ &\quad + F_1(\varphi_1 y, x, z) + F_1(y, x, \varphi_1 z) + F_1(y, \varphi_1 x, \xi_1) \eta_1(z), \end{aligned}$$

$$(13) \quad (\mathfrak{L}_{\xi_1} g)(x, y) = F_1(x, \varphi_1 y, \xi_1) + F_1(y, \varphi_1 x, \xi_1).$$

In [19], it is defined the *associated Nijenhuis tensor* \hat{N}_2 for the almost contact B-metric structure $(\varphi_2, \xi_2, \eta_2)$ by

$$(14) \quad \hat{N}_2 = \{\varphi_2, \varphi_2\} + \xi_2 \otimes \mathfrak{L}_{\xi_2} g,$$

where $\{\varphi_2, \varphi_2\}$ is the symmetric tensor field of type (1, 2) defined as in (10).

Proposition 1. *For the almost contact B-metric manifold $(\mathcal{M}, \varphi_2, \xi_2, \eta_2, g)$, the vanishing of \widehat{N}_2 implies that ξ_2 is Killing.*

Proof. It is known the formula for F_2 in terms of N_2 and \widehat{N}_2 from [9], whereas the expression of \widehat{N}_2 by F_2 is given in [19]:

$$\begin{aligned} F_2(x, y, z) &= -\frac{1}{4} \{ N_2(\varphi_2 x, y, z) + N_2(\varphi_2 x, z, y) \\ &\quad + \widehat{N}_2(\varphi_2 x, y, z) + \widehat{N}_2(\varphi_2 x, z, y) \} \\ &\quad + \frac{1}{2} \eta_2(x) \{ N_2(\xi_2, y, \varphi_2 z) + \widehat{N}_2(\xi_2, y, \varphi_2 z) \\ &\quad + \eta_2(z) \widehat{N}_2(\xi_2, \xi_2, \varphi_2 y) \}, \\ \widehat{N}_2(x, y, z) &= F_2(\varphi_2 x, y, z) - F_2(x, y, \varphi_2 z) + F_2(x, \varphi_2 y, \xi_2) \eta_2(z) \\ &\quad + F_2(\varphi_2 y, x, z) - F_2(y, x, \varphi_2 z) + F_2(y, \varphi_2 x, \xi_2) \eta_2(z). \end{aligned}$$

By the latter equalities, (6) and (7), we obtain the following relation

$$\begin{aligned} (\mathfrak{L}_{\xi_2} g)(x, y) &= -\frac{1}{2} \{ \widehat{N}_2(\varphi_2 x, \varphi_2 y, \xi_2) + \widehat{N}_2(\xi_2, \varphi_2 x, \varphi_2 y) + \widehat{N}_2(\xi_2, \varphi_2 y, \varphi_2 x) \} \\ &\quad + \eta_2(x) \widehat{N}_2(\xi_2, \xi_2, y) + \eta_2(y) \widehat{N}_2(\xi_2, \xi_2, x), \end{aligned}$$

which yields the statement. \square

Let us remark that a similar statement of Proposition 1 for an almost contact metric manifold is not true.

Let the manifold \mathcal{M} be equipped with an almost contact 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $(\alpha = 1, 2, 3)$ and then we consider the product $\mathcal{M} \times \mathbb{R}$. Let X be a vector field on $\mathcal{M} \times \mathbb{R}$ which is presented by a pair $(x, a \frac{d}{dt})$, where x is a tangent vector field on \mathcal{M} , t is the coordinate on \mathbb{R} and a is a differentiable function on $\mathcal{M} \times \mathbb{R}$ [3, Sect. 6.1]. The almost complex structures J_α , $(\alpha = 1, 2, 3)$ are defined on the manifold $\mathcal{M} \times \mathbb{R}$ by

$$(15) \quad J_\alpha X = J_\alpha \left(x, a \frac{d}{dt} \right) = \left(\varphi_\alpha x - a \xi_\alpha, \eta_\alpha(x) \frac{d}{dt} \right).$$

In such a way, in [23], it is defined an almost hypercomplex structure on $\mathcal{M} \times \mathbb{R}$ when \mathcal{M} has an almost contact 3-structure.

Moreover, we equip $\mathcal{M} \times \mathbb{R}$ with the product metric $G = g - dt^2$. By virtue of (15), (2) and its consequence $g(\xi_\alpha, \xi_\alpha) = -\varepsilon_\alpha$, we obtain

$$G(J_\alpha(x, a \frac{d}{dt}), J_\alpha(y, b \frac{d}{dt})) = \varepsilon_\alpha G((x, a \frac{d}{dt}), (y, b \frac{d}{dt})),$$

i.e. the manifold $\mathcal{M} \times \mathbb{R}$ has an almost hypercomplex HN-metric structure (J_α, G) , $(\alpha = 1, 2, 3)$.

We introduce the braces $\{X, Y\}$ for the vector fields $X = (x, a \frac{d}{dt})$ and $Y = (y, b \frac{d}{dt})$ on $\mathcal{M} \times \mathbb{R}$ defined by

$$(16) \quad \{X, Y\} = (\{x, y\}, (x(b) + y(a)) \frac{d}{dt}),$$

where $\{x, y\}$ are given in (8). Obviously, the braces are symmetric.

It is known from [12], the Nijenhuis tensor of two endomorphisms J_α and J_β has the following form:

$$\begin{aligned} 2[J_\alpha, J_\beta](X, Y) &= [J_\alpha X, J_\beta Y] - J_\alpha[J_\beta X, Y] - J_\alpha[X, J_\beta Y] \\ &\quad + [J_\beta X, J_\alpha Y] - J_\beta[J_\alpha X, Y] - J_\beta[X, J_\alpha Y] \\ &\quad + (J_\alpha J_\beta + J_\beta J_\alpha)[X, Y]. \end{aligned}$$

Moreover, the Nijenhuis tensor of an almost complex structure $J_\alpha \equiv J_\beta$ is presented by

$$[J_\alpha, J_\alpha](X, Y) = [J_\alpha X, J_\alpha Y] - J_\alpha[J_\alpha X, Y] - J_\alpha[X, J_\alpha Y] - [X, Y].$$

Analogously of the last two equalities, using the braces (16) instead of the Lie brackets, we define consequently the associated Nijenhuis tensors in the two respective cases as follows:

$$\begin{aligned} 2\{J_\alpha, J_\beta\}(X, Y) &= \{J_\alpha X, J_\beta Y\} - J_\alpha\{J_\beta X, Y\} - J_\alpha\{X, J_\beta Y\} \\ &\quad + \{J_\beta X, J_\alpha Y\} - J_\beta\{J_\alpha X, Y\} - J_\beta\{X, J_\alpha Y\} \\ &\quad + (J_\alpha J_\beta + J_\beta J_\alpha)\{X, Y\}, \\ (17) \quad \{J_\alpha, J_\alpha\}(X, Y) &= \{J_\alpha X, J_\alpha Y\} - J_\alpha\{J_\alpha X, Y\} - J_\alpha\{X, J_\alpha Y\} \\ &\quad - \{X, Y\}. \end{aligned}$$

The latter tensor is given in [15] and coincides with the tensor \tilde{N} introduced in [5] by an equivalent equality of (17).

According to [7], the \mathcal{G}_1 -manifolds are almost Hermitian manifolds whose corresponding Nijenhuis (0,3)-tensor by the Hermitian metric is a 3-form. This condition is equivalent to the vanishing of the associated Nijenhuis tensor, according to [16].

As it is known from [5], the class of the quasi-Kähler manifolds with Norden metric is the only basic class of the considered manifolds with non-integrable almost complex structure J , because $[J, J]$ is non-zero there. Moreover, this class is determined by the condition $\{J, J\} = 0$.

In [16], it is proven the following

Proposition 2. *Let (J_1, J_2, J_3) be an almost hypercomplex structure and G is a pseudo-Riemannian metric on the almost hypercomplex manifold. If two of its six associated Nijenhuis tensors $\{J_1, J_1\}$, $\{J_2, J_2\}$, $\{J_3, J_3\}$, $\{J_1, J_2\}$, $\{J_1, J_3\}$, $\{J_2, J_3\}$ vanish, then the others also vanish.*

We seek to express in terms of the structure tensors of $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$ a necessary and sufficient condition for $\{J_\alpha, J_\alpha\} = 0$.

For the structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, 2, 3\}$, let us define the following four tensors of type (1,2), (0,2), (1,1), (0,1), respectively:

$$\begin{aligned} \hat{N}_\alpha^{(1)}(x, y) &= \{\varphi_\alpha, \varphi_\alpha\}(x, y) - \varepsilon_\alpha(\mathfrak{L}_{\xi_\alpha} g)(x, y) \cdot \xi_\alpha, \\ \hat{N}_\alpha^{(2)}(x, y) &= -\varepsilon_\alpha(\mathfrak{L}_{\xi_\alpha} g)(\varphi_\alpha x, y) - \varepsilon_\alpha(\mathfrak{L}_{\xi_\alpha} g)(x, \varphi_\alpha y), \\ \hat{N}_\alpha^{(3)}(x) &= \{\varphi_\alpha, \varphi_\alpha\}(\varphi_\alpha x, \xi_\alpha) + (\mathfrak{L}_{\xi_\alpha} \eta_\alpha)(\varphi_\alpha x) \cdot \xi_\alpha + 2\eta_\alpha(x)\varphi_\alpha \nabla_{\xi_\alpha} \xi_\alpha, \\ \hat{N}_\alpha^{(4)}(x) &= -(\mathfrak{L}_{\xi_\alpha} \eta_\alpha)(x). \end{aligned} \quad (18)$$

Proposition 3. *The associated Nijenhuis tensor $\{J_\alpha, J_\alpha\}$ of an almost complex structure J_α for some $(\mathcal{M} \times \mathbb{R}, J_\alpha, G)$, $\alpha \in \{1, 2, 3\}$, vanishes if and only if the four tensors $\hat{N}_\alpha^{(1)}$, $\hat{N}_\alpha^{(2)}$, $\hat{N}_\alpha^{(3)}$, $\hat{N}_\alpha^{(4)}$ for the structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ vanish.*

Proof. First of all we need of the following relations

$$(19) \quad (\mathfrak{L}_{\xi_\alpha} g)(\xi_\alpha, x) = -\varepsilon_\alpha(\mathfrak{L}_{\xi_\alpha} \eta_\alpha)(x) = g(\nabla_{\xi_\alpha} \xi_\alpha, x).$$

These equalities follow by virtue of (5), (6), (7).

Since any $\{J_\alpha, J_\alpha\}$ is a tensor field of type $(1, 2)$, it suffices to compute the tensors $\{J_\alpha, J_\alpha\}((x, 0), (y, 0))$ and $\{J_\alpha, J_\alpha\}((x, 0), (o, \frac{d}{dt}))$, where o is the zero element of $\mathfrak{X}(\mathcal{M})$. Taking into account (10), (15), (16), (17), we obtain consequently:

$$\begin{aligned} \{J_\alpha, J_\alpha\}((x, 0\frac{d}{dt}), (y, 0\frac{d}{dt})) &= \\ &= \{(\varphi_\alpha x, \eta_\alpha(x)\frac{d}{dt}), (\varphi_\alpha y, \eta_\alpha(y)\frac{d}{dt})\} - \{(x, 0\frac{d}{dt}), (y, 0\frac{d}{dt})\} \\ &\quad - J_\alpha\{(\varphi_\alpha x, \eta_\alpha(x)\frac{d}{dt}), (y, 0\frac{d}{dt})\} - J_\alpha\{(x, 0\frac{d}{dt}), (\varphi_\alpha y, \eta_\alpha(y)\frac{d}{dt})\} \\ &= (\{\varphi_\alpha x, \varphi_\alpha y\}, (\varphi_\alpha x(\eta_\alpha(y)) + \varphi_\alpha y(\eta_\alpha(x)))\frac{d}{dt}) \\ &\quad - (-\varphi_\alpha^2\{x, y\} + \eta_\alpha(\{x, y\})\xi_\alpha, 0\frac{d}{dt}) \\ &\quad - (\varphi_\alpha\{\varphi_\alpha x, y\} - y(\eta_\alpha(x))\xi_\alpha, \eta_\alpha(\{\varphi_\alpha x, y\})\frac{d}{dt}) \\ &\quad - (\varphi_\alpha\{x, \varphi_\alpha y\} - x(\eta_\alpha(y))\xi_\alpha, \eta_\alpha(\{x, \varphi_\alpha y\})\frac{d}{dt}) \\ &= (\widehat{N}_\alpha^{(1)}(x, y), \widehat{N}_\alpha^{(2)}(x, y)\frac{d}{dt}), \\ \{J_\alpha, J_\alpha\}((x, 0\frac{d}{dt}), (o, \frac{d}{dt})) &= \\ &= \{(\varphi_\alpha x, \eta_\alpha(x)\frac{d}{dt}), (-\xi_\alpha, 0\frac{d}{dt})\} - \{(x, 0\frac{d}{dt}), (o, \frac{d}{dt})\} \\ &\quad - J_\alpha\{(\varphi_\alpha x, \eta_\alpha(x)\frac{d}{dt}), (o, \frac{d}{dt})\} - J_\alpha\{(x, 0\frac{d}{dt}), (-\xi_\alpha, 0\frac{d}{dt})\} \\ &= -(\{\varphi_\alpha x, \xi_\alpha\}, \xi_\alpha(\eta_\alpha(x))\frac{d}{dt}) + (\varphi_\alpha\{x, \xi_\alpha\}, \eta_\alpha(\{x, \xi_\alpha\})\frac{d}{dt}) \\ &= (\widehat{N}_\alpha^{(3)}x, \widehat{N}_\alpha^{(4)}(x)\frac{d}{dt}). \end{aligned}$$

Then, for any $\alpha = 1, 2, 3$, the vanishing of $\{J_\alpha, J_\alpha\}$ holds if and only if $\widehat{N}_\alpha^{(1)}, \widehat{N}_\alpha^{(2)}, \widehat{N}_\alpha^{(3)}, \widehat{N}_\alpha^{(4)}$ vanish. \square

Proposition 4. *For an almost contact structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1, 2, 3\}$ and a pseudo-Riemannian metric g , the vanishing of $\widehat{N}_\alpha^{(1)}$ implies the vanishing of $\widehat{N}_\alpha^{(2)}, \widehat{N}_\alpha^{(3)}$ and $\widehat{N}_\alpha^{(4)}$.*

Proof. We set $y = \xi_\alpha$ in $\widehat{N}_\alpha^{(1)}(x, y) = 0$ and apply η_α . Then, using (10) and (1), we obtain $(\mathfrak{L}_{\xi_\alpha} g)(x, \xi_\alpha) = 0$ and thus $\widehat{N}_\alpha^{(4)} = 0$, according to (19).

Therefore, from the form of $\widehat{N}_\alpha^{(1)}$ in (18), we get $\{\varphi_\alpha, \varphi_\alpha\}(\varphi_\alpha x, \xi_\alpha) = 0$. On the other hand, bearing in mind (19), we have that the vanishing of $(\mathfrak{L}_{\xi_\alpha} g)(x, \xi_\alpha)$ is equivalent to the vanishing of $(\mathfrak{L}_{\xi_\alpha} \eta_\alpha)(x)$ and $\nabla_{\xi_\alpha} \xi_\alpha$. Thus, we obtain $\widehat{N}_\alpha^{(3)} = 0$.

Finally, applying η_α to $\widehat{N}_\alpha^{(1)}(\varphi_\alpha x, y) = 0$ and using (10), we have

$$\eta_\alpha(\{\varphi_\alpha^2 x, \varphi_\alpha y\}) - \varepsilon_\alpha(\mathfrak{L}_{\xi_\alpha} g)(\varphi_\alpha x, y) = 0.$$

The first term in the latter equality can be expressed in the following form

$$-\varepsilon_\alpha(\mathfrak{L}_{\xi_\alpha} g)(x, \varphi_\alpha y),$$

using that $\mathfrak{L}_{\xi_\alpha} \eta_\alpha$ vanishes. In such a way we obtain that $\widehat{N}_\alpha^{(2)}(x, y) = 0$. \square

Proposition 5. *For an almost contact structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1, 2, 3\}$ and a pseudo-Riemannian metric g , where ξ_α is Killing, $\widehat{N}_\alpha^{(2)}$ and $\widehat{N}_\alpha^{(4)}$ vanish. Moreover, we have the following:*

- (i) $\widehat{N}_\alpha^{(1)}$ vanishes if and only if $\{\varphi_\alpha, \varphi_\alpha\}$ vanishes;
- (ii) $\widehat{N}_\alpha^{(3)}$ vanishes if and only if $\xi_\alpha \lrcorner \{\varphi_\alpha, \varphi_\alpha\}$ vanishes.

Proof. Taking into account that $\mathfrak{L}_{\xi_\alpha} g$ vanishes, we have $\widehat{N}_\alpha^{(1)} = \{\varphi_\alpha, \varphi_\alpha\}$ and $\widehat{N}_\alpha^{(2)} = 0$. Further, we obtain $\widehat{N}_\alpha^{(4)} = 0$ and $\widehat{N}_\alpha^{(3)} x = \{\varphi_\alpha, \varphi_\alpha\}(\varphi_\alpha x, \xi_\alpha)$, according to (19). Then, (i) is obvious whereas (ii) holds, bearing in mind the assumption for ξ_α . \square

Let $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $(\alpha = 1, 2, 3)$ be a manifold with almost contact HN-metric 3-structure. The symmetric $(1, 2)$ -tensors defined by

$$(20) \quad \widehat{N}_\alpha = \{\varphi_\alpha, \varphi_\alpha\} - \varepsilon_\alpha \xi_\alpha \otimes \mathfrak{L}_{\xi_\alpha} g$$

we call *associated Nijenhuis tensors* on $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$.

The corresponding $(0, 3)$ -tensors are denoted by

$$\widehat{N}_\alpha(x, y, z) = g(\widehat{N}_\alpha(x, y), z), \quad \{\varphi_\alpha, \varphi_\alpha\}(x, y, z) = g(\{\varphi_\alpha, \varphi_\alpha\}(x, y), z).$$

Then, taking into account (5) and (20), we obtain

$$\widehat{N}_\alpha(x, y, z) = \{\varphi_\alpha, \varphi_\alpha\}(x, y, z) + (\mathfrak{L}_{\xi_\alpha} g)(x, y) \eta_\alpha(z).$$

Theorem 6. *Let $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $(\alpha = 1, 2, 3)$ be a manifold with almost contact HN-metric 3-structure. For any α , the associated Nijenhuis tensor $\{J_\alpha, J_\alpha\}$ of the almost complex structure J_α on $(\mathcal{M} \times \mathbb{R}, J_\alpha, G)$ vanishes if and only if the associated Nijenhuis tensor \widehat{N}_α of the structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ vanishes.*

Proof. The statement follows from Proposition 3 and Proposition 4, bearing in mind (18) and (20). \square

Theorem 7. *Let $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $(\alpha = 1, 2, 3)$ be a manifold with almost contact HN-metric 3-structure. If two of the associated Nijenhuis tensors \widehat{N}_α vanish, the third also vanishes.*

Proof. It follows by virtue of Proposition 2 and Theorem 6. \square

4. NATURAL CONNECTIONS WITH TOTALLY SKEW-SYMMETRIC TORSION

A linear connection D is said to be a *natural connection* for $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, 2, 3\}$, if it preserves the structure, i.e.

$$D\varphi_\alpha = D\xi_\alpha = D\eta_\alpha = Dg = 0.$$

Theorem 8. *Let $(\mathcal{M}, \varphi_1, \xi_1, \eta_1, g)$ be a pseudo-Riemannian manifold with an almost contact metric structure. The following statements are equivalent:*

- (i) *The manifold belongs to the class $\mathcal{W}_2 \oplus \mathcal{W}_4 \oplus \mathcal{W}_9 \oplus \mathcal{W}_{10} \oplus \mathcal{W}_{11}$ determined by*
- $$(21) \quad F_1(\varphi_1 x, y, z) + F_1(\varphi_1 y, x, z) + F_1(x, y, \varphi_1 z) + F_1(y, x, \varphi_1 z) = 0.$$
- (ii) *The associated Nijenhuis tensor \widehat{N}_1 vanishes and ξ_1 is a Killing vector field;*
 - (iii) *The tensor $\{\varphi_1, \varphi_1\}$ vanishes and ξ_1 is a Killing vector field;*
 - (iv) *The Nijenhuis tensor N_1 is a 3-form and ξ_1 is a Killing vector field;*
 - (v) *There exists a natural connection D^1 with totally skew-symmetric torsion for the structure $(\varphi_1, \xi_1, \eta_1, g)$ and this connection is unique.*

Proof. Using (12) and (13), we have that the vanishing of \widehat{N}_1 and $\mathfrak{L}_{\xi_1}g$ implies the identity (21). Viceversa, setting $x = y = \xi_1$ in (21), we have $F_1(\xi_1, \xi_1, z) = 0$. If we put $x = \varphi_1 x$, $y = \varphi_1 y$, $z = \xi_1$ in (21) and use the latter vanishing, we obtain that $\mathfrak{L}_{\xi_1}g = 0$ and therefore $\widehat{N}_1 = 0$. The determination of the class in (i) by (21) follows immediately from the definition of the basic classes of the classification in [2]. So, the equivalence between (i) and (ii) is valid.

Now, we need to prove the following relation

$$(22) \quad \widehat{N}_1(x, y, z) = N_1(z, x, y) + N_1(z, y, x).$$

We calculate the right hand side of (22) using (11). Taking into account (4) and their consequence

$$(23) \quad \begin{aligned} F_1(x, y, \varphi_1 z) &= F_1(x, \varphi_1 y, z) + F_1(x, \xi_1, \varphi_1 y)\eta_1(z) \\ &\quad + F_1(x, \xi_1, \varphi_1 z)\eta_1(y), \end{aligned}$$

we obtain

$$\begin{aligned} N_1(z, x, y) + N_1(z, y, x) &= -F_1(\varphi_1 x, z, y) - F_1(\varphi_1 y, z, x) \\ &\quad - F_1(x, z, \varphi_1 y) - F_1(y, z, \varphi_1 x) \\ &\quad - F_1(x, \varphi_1 z, \xi_1)\eta_1(y) - F_1(y, \varphi_1 z, \xi_1)\eta_1(x). \end{aligned}$$

Using again (23) and the first equality in (4), we establish that the right hand side of the latter equality is equal to $\widehat{N}_1(x, y, z)$, according to (12). Therefore, (22) is valid.

The relation (22) implies the equivalence between (ii) and (iv), whereas the equivalence between (iv) and (v) is given in Theorem 8.2 of [4]. The equivalence between (ii) and (iii) follows from (9). \square

For the natural connection D^1 with totally skew-symmetric torsion for the structure $(\varphi_1, \xi_1, \eta_1, g)$, we have

$$(24) \quad g(D_x^1 y, z) = g(\nabla_x y, z) + \frac{1}{2}T_1(x, y, z)$$

and its torsion T_1 , according to Theorem 8.2 of [4], is determined in our notations by

$$(25) \quad T_1 = -\eta_1 \wedge d\eta_1 + d_1^\varphi \Phi + N_1 - \eta_1 \wedge (\xi_1 \lrcorner N_1),$$

where it is used the notation $d^\varphi \Phi(x, y, z) = -d\Phi(\varphi_1 x, \varphi_1 y, \varphi_1 z)$ for the fundamental 2-form Φ of the almost contact metric structure, i.e. $\Phi(x, y) = g(x, \varphi_1 y)$.

Since $\eta_1 \wedge d\eta_1 = \mathfrak{S}\{\eta_1 \otimes d\eta_1\}$ holds and because of (4), (6) and the fact that ξ_1 is Killing, it is valid the following

$$(26) \quad (\eta_1 \wedge d\eta_1)(x, y, z) = -2 \mathfrak{S}_{x, y, z} \{\eta_1(x)F_1(y, \varphi_1 z, \xi_1)\}.$$

Moreover, from the equalities $d\Phi(x, y, z) = - \mathfrak{S}_{x, y, z} F_1(x, y, z)$ and (4), we get

$$(27) \quad d^\varphi \Phi(x, y, z) = - \mathfrak{S}_{x, y, z} \{F_1(\varphi_1 x, y, z) + 2F_1(x, \varphi_1 y, \xi_1)\eta_1(z)\}.$$

So, applying (26), (27), (11) and (4) to the equality (25), we obtain an expression of T_1 in terms of F_1 as follows

$$(28) \quad \begin{aligned} T_1(x, y, z) &= F_1(x, y, \varphi_1 z) - F_1(y, x, \varphi_1 z) - F_1(\varphi_1 z, x, y) \\ &\quad + 2F_1(x, \varphi_1 y, \xi_1)\eta_1(z). \end{aligned}$$

The equivalences in the following theorem are known from [14] and [19].

Theorem 9. *The following statements for an almost contact B-metric manifold $(\mathcal{M}, \varphi_2, \xi_2, \eta_2, g)$ are equivalent:*

- (i) *It belongs to the class $\mathcal{F}_3 \oplus \mathcal{F}_7$, which is characterised by the conditions: the cyclic sum of F_2 by the three arguments vanishes and ξ_2 is Killing;*
- (ii) *It has a vanishing associated Nijenhuis tensor \hat{N}_2 ;*
- (iii) *It has a vanishing tensor $\{\varphi_2, \varphi_2\}$ and ξ_2 is Killing;*
- (iv) *It admits the existence of a unique natural connection D^2 with totally skew-symmetric torsion.*

Proof. The equivalence of (i), (ii) and (iv) is known from [14] and [19], whereas the equivalence of (ii) and (iii) follows from (14) and Proposition 1. \square

For the natural connection D^2 with totally skew-symmetric torsion for the structure $(\varphi_2, \xi_2, \eta_2, g)$, we have

$$(29) \quad g(D_x^2 y, z) = g(\nabla_x y, z) + \frac{1}{2} T_2(x, y, z),$$

where its torsion T_2 is determined by $T_2 = \eta_2 \wedge d\eta_2 + \frac{1}{4} \mathfrak{S} N_2$ and it is expressed in terms of F_2 by

$$(30) \quad T_2(x, y, z) = -\frac{1}{2} \mathfrak{S}_{x,y,z} \{F_2(x, y, \varphi_2 z) - 3\eta_2(x)F_2(y, \varphi_2 z, \xi_2)\}.$$

Using Theorem 8, Theorem 9 and Proposition 1, we get immediately the following

Theorem 10. *Let $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $(\alpha = 1, 2, 3)$ be a manifold with almost contact HN-metric 3-structure. The existence of unique natural connections with totally skew-symmetric torsion for two of the three structures implies an existence of a unique natural connection with totally skew-symmetric torsion for the remaining third structure.*

Corollary 11. *Let $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $(\alpha = 1, 2, 3)$ be a manifold with almost contact HN-metric 3-structure. If the manifold belongs to two of the following three classes for the corresponding structure, then the manifold belongs to the remaining third class for the corresponding structure:*

- (i) $\mathcal{W}_2 \oplus \mathcal{W}_4 \oplus \mathcal{W}_9 \oplus \mathcal{W}_{10} \oplus \mathcal{W}_{11}$ for $\alpha = 1$;
- (ii) $\mathcal{F}_3 \oplus \mathcal{F}_7$ for $\alpha = 2$;
- (iii) $\mathcal{F}_3 \oplus \mathcal{F}_7$ for $\alpha = 3$.

Now, we are interested on conditions for coincidence of these three natural connections D^α , $(\alpha = 1, 2, 3)$ with totally skew-symmetric torsion for the particular almost contact structures with the metric g . Then we shall say that it exists a natural connection with totally skew-symmetric torsion for the almost contact HN-metric 3-structure.

Theorem 12. *Let $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $(\alpha = 1, 2, 3)$ be a manifold with almost contact HN-metric 3-structure for which the associated Nijenhuis tensors \hat{N}_α vanish and ξ_1 is Killing. This manifold admits a linear connection D with totally skew-symmetric torsion preserving the almost contact HN-metric 3-structure if and only*

if the following equalities are valid

$$\begin{aligned}
 & F_1(x, y, \varphi_1 z) - F_1(y, x, \varphi_1 z) - F_1(\varphi_1 z, x, y) + 2F_1(x, \varphi_1 y, \xi_1)\eta_1(z) \\
 (31) \quad &= -\frac{1}{2} \underset{x, y, z}{\mathfrak{S}} \{F_2(x, y, \varphi_2 z) - 3\eta_2(x)F_2(y, \varphi_2 z, \xi_2)\} \\
 &= -\frac{1}{2} \underset{x, y, z}{\mathfrak{S}} \{F_2(x, y, \varphi_3 z) - 3\eta_3(x)F_3(y, \varphi_3 z, \xi_3)\}.
 \end{aligned}$$

If D exists, it is unique.

Proof. According to Theorem 8 and Theorem 9, since $\hat{N}_\alpha = \mathfrak{L}_{\xi_1}g = 0$ are valid then there exist the natural connections D^α , ($\alpha = 1, 2, 3$) with totally skew-symmetric torsion T_α for the structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$. Bearing in mind (24), (28), (29) and (30), the coincidence of D^1 , D^2 and D^3 is equivalent to the conditions (31). \square

5. A 7-DIMENSIONAL LIE GROUP AS A MANIFOLD WITH ALMOST CONTACT HN-METRIC 3-STRUCTURE

Let \mathcal{L} be a 7-dimensional real connected Lie group, and \mathfrak{l} be its Lie algebra with a basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Then an arbitrary vector x in $T_p\mathcal{L}$ at $p \in \mathcal{L}$ is presented by $x = x^i e_i$ ($i = 1, 2, \dots, 7$).

Now we introduce an almost contact HN-metric 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$ by a standard way as follows

$$\begin{aligned}
 (32) \quad & \begin{aligned}
 \varphi_1 e_1 &= e_2, & \varphi_1 e_2 &= -e_1, & \varphi_1 e_3 &= e_4, & \varphi_1 e_4 &= -e_3, \\
 \varphi_1 e_5 &= o, & \varphi_1 e_6 &= e_7, & \varphi_1 e_7 &= -e_6, & & \\
 \varphi_2 e_1 &= e_3, & \varphi_2 e_2 &= -e_4, & \varphi_2 e_3 &= -e_1, & \varphi_2 e_4 &= e_2, \\
 \varphi_2 e_5 &= -e_7, & \varphi_2 e_6 &= o, & \varphi_2 e_7 &= e_5, & & \\
 \varphi_3 e_1 &= e_4, & \varphi_3 e_2 &= e_3, & \varphi_3 e_3 &= -e_2, & \varphi_3 e_4 &= -e_1, \\
 \varphi_3 e_5 &= e_6, & \varphi_3 e_6 &= -e_5, & \varphi_3 e_7 &= o, & & \\
 \xi_1 &= e_5, & \xi_2 &= e_6, & \xi_3 &= e_7, & & \\
 \eta_1 &= x^5, & \eta_2 &= x^6, & \eta_3 &= x^7, & &
 \end{aligned}
 \end{aligned}$$

where o is the zero vector in $T_p\mathcal{L}$, $p \in \mathcal{L}$.

Let g be a pseudo-Riemannian metric such that

$$\begin{aligned}
 g(e_1, e_1) &= g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) \\
 &= -g(e_5, e_5) = g(e_6, e_6) = g(e_7, e_7) = 1, \\
 g(e_i, e_j) &= 0, \quad i \neq j.
 \end{aligned}$$

The almost contact HN-metric 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$ on H coincides with the almost hypercomplex HN-metric structure considered in [8]. The almost hypercomplex structure is defined as in [20].

Let us consider $(\mathcal{L}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ with the Lie algebra \mathfrak{l} determined by the following nonzero commutators:

$$[e_1, e_2] = [e_3, e_4] = \lambda e_7,$$

where $\lambda \in \mathbb{R} \setminus \{0\}$.

By the well-known Koszul equality, we compute the components of the Levi-Civita connection ∇ with respect to the basis and the nonzero ones of them are:

$$\begin{aligned}
 (33) \quad & \begin{aligned}
 \nabla_{e_1} e_2 &= -\nabla_{e_2} e_1 = \nabla_{e_3} e_4 = -\nabla_{e_4} e_3 = \frac{1}{2} \lambda e_7, \\
 \nabla_{e_1} e_7 &= \nabla_{e_7} e_1 = -\frac{1}{2} \lambda e_2, & \nabla_{e_2} e_7 &= \nabla_{e_7} e_2 = \frac{1}{2} \lambda e_1, \\
 \nabla_{e_3} e_7 &= \nabla_{e_7} e_3 = \frac{1}{2} \lambda e_4, & \nabla_{e_4} e_7 &= \nabla_{e_7} e_4 = -\frac{1}{2} \lambda e_3.
 \end{aligned}
 \end{aligned}$$

Proposition 13. *Let $(\mathcal{L}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $(\alpha = 1, 2, 3)$, be the Lie group \mathcal{L} with almost contact HN-metric 3-structure depending on the nonzero real parameter λ . Then this manifold belongs to the following basic classes, according to the corresponding classification in [2] and [6]:*

- \mathcal{W}_{10} with respect to $(\varphi_1, \xi_1, \eta_1, g)$;
- \mathcal{F}_3 with respect to $(\varphi_2, \xi_2, \eta_2, g)$;
- \mathcal{F}_7 with respect to $(\varphi_3, \xi_3, \eta_3, g)$.

Proof. Using (3), (32) and (33), we obtain the basic components of tensors $(F_\alpha)_{ijk} = F_\alpha(e_i, e_j, e_k)$ as follows:

$$\begin{aligned}
 (34) \quad \frac{1}{2}\lambda &= (F_1)_{117} = (F_1)_{126} = -(F_1)_{216} = (F_1)_{227} \\
 &= (F_1)_{337} = (F_1)_{346} = -(F_1)_{436} = (F_1)_{447} \\
 &= (F_2)_{125} = (F_2)_{147} = -(F_2)_{215} = (F_2)_{237} \\
 &= -(F_2)_{327} = (F_2)_{345} = -(F_2)_{417} = -(F_2)_{435} \\
 &= -(F_3)_{137} = (F_3)_{247} = (F_3)_{317} = -(F_3)_{427}.
 \end{aligned}$$

From (34), applying the classification conditions for the relevant classification in [2] or [6], we have the classes in the statement, respectively. \square

Bearing in mind Proposition 13, we deduce that the conditions (i) of Theorem 8 and Theorem 9 are fulfilled and therefore there exist natural connections D^α ($\alpha = 1, 2, 3$) for the corresponding structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ on \mathcal{L} . We get the components with respect to the basis of their torsions T_α ($\alpha = 1, 2, 3$) by direct computations from (24), (28), (29), (30) and (34) as follows

$$\begin{aligned}
 (T_1)_{127} &= (T_1)_{347} = -\lambda, \\
 (T_2)_{127} &= -(T_2)_{145} = -(T_2)_{235} = (T_2)_{347} = -\frac{1}{2}\lambda, \\
 (T_3)_{127} &= (T_3)_{347} = -\lambda.
 \end{aligned}$$

Obviously, the connections D^1 and D^3 coincides but D^2 differs from them. The condition (31) of Theorem 12 is not fulfilled and therefore it does not exist a unique connection with totally skew-symmetric torsion preserving the almost contact HN-metric 3-structure on \mathcal{L} .

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REFERENCES

- [1] D. V. Alekseevsky and S. Marchiafava, Quaternionic structures on a manifold and subordinated structures, Ann. Mat. Pura Appl. (IV) **CLXXI** (1996), 205-273.
- [2] V. Alexiev and G. Ganchev, On the classification of the almost contact metric manifolds, Math. Educ. Math., Proc. 15th Spring Conf. of UBM, Sunny Beach, 1986, 155-161. (English translation in arXiv:1110.4297)
- [3] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics 203, Birkhäuser, Boston, 2002.
- [4] T. Friedrich and S. Ivanov, Parallel spinors and connections with skew-symmetric torsion in string theory, Asian J. Math. **6** (2002), 303-336.
- [5] G. Ganchev and A. Borisov, Note on the almost complex manifolds with a Norden metric, C. R. Acad. Bulgare Sci. **39** (5) (1986), 31-34.
- [6] G. Ganchev, V. Mihova and K. Gribachev, Almost contact manifolds with B-metric, Math. Balkanica (N.S.) **7** (1993), 261-276.
- [7] A. Gray and L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl. (IV) **CXXXIII** (1980), 35-58.

- [8] K. Gribachev, M. Manev and S. Dimiev, On the almost hypercomplex pseudo-Hermitian manifolds, in: Trends in Complex Analysis, Differential Geometry and Mathematical Physics, eds. S. Dimiev and K. Sekigawa, 51-62, World Sci. Publ., Singapore, River Edge, NJ, 2003.
- [9] S. Ivanov, H. Manev and M. Manev, Sasaki-like almost contact complex Riemannian manifolds, arXiv:1402.5426.
- [10] Y.-Y. Kuo, On almost contact 3-structure, Tôhoku Math. J. (2), **22** (1970), no. 3, 325-332.
- [11] K. Gribachev and M. Manev, Almost hypercomplex pseudo-Hermitian manifolds and a 4-dimensional Lie group with such structure, J. Geom. **88**, no. 1-2, (2008), 41-52.
- [12] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. II. Interscience Tracts in Pure and Applied Mathematics, No. 15, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.
- [13] M. Manev, A connection with parallel torsion on almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics, J. Geom. Phys. **61** (2011), no. 1, 248-259.
- [14] M. Manev, Natural connection with totally skew-symmetric torsion on almost contact manifolds with B-metric, Int. J. Geom. Methods Mod. Phys. **9** (2012), no. 5, 1250044 (20 pages).
- [15] M. Manev, On canonical-type connections on almost contact complex Riemannian manifolds, Filomat **29** (2015), no. 3, 411-425.
- [16] M. Manev, Associated Nijenhuis tensors on manifolds with almost hypercomplex structures and metrics of Hermitian-Norden type, arXiv:1510.00821.
- [17] M. Manev, Manifolds with Almost Contact 3-Structure and Metrics of Hermitian-Norden Type, arXiv:1506.04376.
- [18] M. Manev and K. Gribachev, A connection with parallel totally skew-symmetric torsion on a class of almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics, Int. J. Geom. Methods Mod. Phys. **8** (2011), no. 1, 115-131.
- [19] M. Manev and M. Ivanova, A classification of the torsion tensors on almost contact manifolds with B-metric, Cent. Eur. J. Math. **12** (2014), no. 10, 1416-1432.
- [20] A. Sommese, Quaternionic manifolds, Math. Ann. **212** (1975), 191-214.
- [21] C. Udriște, Structures presque coquaternioniennes, Bull. Math. Soc. Sci. Math. R. S. Roumanie **13** (1969), 487-507.
- [22] K. Yano and M. Ako, Intergrability conditions for almost quaternionic structures, Hokkaido Math. J. **1** (1972), 63-86.
- [23] K. Yano, S. Ishihara and M. Konishi, Normality of almost contact 3-structure, Tôhoku Math. J. **25** (1973), 167-175.

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